On the attenuation of long gravity waves by short breaking waves

By O. M. PHILLIPS
Department of Applied Mathematics and Theoretical Physics, University of Cambridge

(Received 14 January 1963)

It is shown that when short wind-generated gravity waves lose energy by breaking (or other dissipative processes) near the crests of longer waves, the loss is supplied partly by the longer wave because of the second-order radiation-stress interaction. This process is discussed in detail analytically and also from energy considerations with use of the concept of radiation stress. The results are applied to the attenuation of swell by a local wind-generated wave field, and it is shown that the rate of decrease of the amplitude a'' of the swell is constant and given by

$$\frac{da''}{dt} = -\frac{3\rho_a u_*^2 c}{2\rho_w c''^2},$$

where ρ_a, ρ_w are the air and water densities, u_* is the friction velocity of the wind in the interaction zone, and c'', c are the phase velocities of the swell and the component of the locally generated field at the spectral maximum.

1. Introduction

This paper is concerned with the dynamical consequences of the breaking of short-gravity waves near the crests of longer waves. In the open sea, an observer can discern wave breaking of at least two types. One, with which we will not be concerned, may occur when the crests of two low-frequency waves intersect or overtake each other; this is characterized by a tumbling of the water and the vigorous entrainment of air bubbles, giving rise to the name whitecaps or white horses. The second kind occurs when short wind-generated gravity waves are overtaken by a much longer wave. Near the crests of the long wave, the energy density of the short wave is increased partly by the contraction associated with the long wave and partly by the interaction that Longuet-Higgins & Stewart (1960) showed could be described in terms of a radiation stress. If the short waves are receiving energy from the wind, their amplitude will grow until at the long wave crests they will approach their limiting configuration allowed by the stability requirements of the surface. The short-wave crests become sharp, and the waves may break in the usual sense, or splash when different short waves interfere. Another possibility is that, as sharp crests appear on the short waves, capillary waves develop on the forward face resulting, as Longuet-Higgins (1963) recently showed, in a much enhanced dissipation locally from the short waves. In any event, energy is lost from the short waves near the crests of the long ones. It will be shown later than this energy lost from the short waves is supplied not only by the wind but also, through the interaction represented by the radiation stress, by the longer waves. As a consequence, the 'breaking' of short waves at the crest of longer ones (using this term to include any of the energy-loss mechanisms) results in an energy loss from the long waves and, unless the wind can make up this loss, in an attenuation.

This process then represents a way in which energy can be extracted from long waves by interaction with short ones. The process is irreversible and the energy is lost from the wave field: it does not represent a genuine energy flux from one component of the wave field to another. It is known (Phillips 1960; Hasselmann 1962, 1963) that energy transfer of the latter kind occurs only at the third order in a Stokes expansion, whereas the radiation stresses are second-order phenomena. In the absence of short-wave 'breaking' (in this sense) the effect of the radiation stresses associated with the interaction is to cause the short-wave energy density to oscillate with the frequency of the long wave. There is no net energy transfer, but only a continual oscillatory interchange. If the short waves lose energy where their energy density is highest (at the long wave crests), part of the energy lost is that which has been 'borrowed' from the long waves, so that as time goes on, they will attenuate.

The dynamical consequences of this are first discussed by a direct study of the non-linear interaction between the two wave trains. The results are interpreted physically in terms of the radiation-stress concept and applied to the case of the attenuation of swell passing through a local wind-generated sea.

2. Analysis of the interaction

If waves are generated on the surface of a real fluid by normal stress fluctuations, it is known that a second-order vorticity diffuses downwards from the free surface. Although this vorticity distribution is of crucial importance in questions of mass transport (Longuet-Higgins 1953), it does not affect the dynamics of the wave motion to second order. We can consequently suppose the wave motion to be irrotational, so that the velocity fluid is given by

$$\mathbf{u} = \nabla\phi \quad \text{and} \quad \nabla^2\phi = 0. \tag{2.1}$$

The boundary conditions to be satisfied on the free surface $z = \xi(x,t)$ are

$$\frac{\partial\xi}{\partial t} + \frac{\partial\xi}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)_\xi - \left(\frac{\partial\phi}{\partial z}\right)_\xi = 0, \tag{2.2}$$

$$\frac{p_0}{\rho} + g\xi + \tfrac{1}{2}(u^2)_\xi + \left(\frac{\partial\phi}{\partial t}\right)_\xi = 0. \tag{2.3}$$

The first of these is the kinematic boundary condition, and the second the condition of continuity of pressure across the free surface, where $p_0(x,t)$ represents the pressure distribution on the surface set up by the air flow over the water. In addition, in deep water we have the condition that $\phi \to 0$ as $z \to -\infty$.

Now, the turbulent wind flow over a single wave train moving in the positive x-direction sets up a component of the pressure distribution in phase with the wave slope that can be represented as

$$p_1(x,t) = \gamma\rho c^2 \partial\xi/\partial x, \tag{2.4}$$

where c is the phase speed of the waves and ρ the water density. The numerical value of γ is determined by the velocity profile in the wind, and analyses of the generation of waves by the instability mechanism (Miles 1957, 1960; Brooke Benjamin 1959; Lighthill 1962) are concerned largely with estimating γ. It is seen below that $(\pi\gamma)^{-1}$ represents the number of wave periods required for the wave amplitude to increase by a factor e under the action of the instability and in the absence of any damping. The numerical value of γ is of order 10^{-2} for the short-gravity wave components of a wind-generated wave field (those for which $c \sim u_*$, the wind friction velocity), but for the longer components γ is very much less. For example, Phillips & Katz (1961) showed that $\gamma \sim 10^{-4}$ when $c \sim 8u_*$. In the present problem, we are interested in the interaction between short waves and much longer ones; the growth of the long waves over one wave period is very small whereas over the same time interval the short-waves can grow significantly since this represents a considerable number of short wave periods and also because γ is numerically larger for the short waves. Consequently, it is reasonable to neglect the direct wind action on the longer waves and to consider the interaction between growing short waves and long waves of fixed amplitude.

It should be remembered that the wind flow over the water also induces a pressure variation in phase with the surface displacement which is numerically much larger than that given by (2.4). However, this component has a negligible influence on the energy balance of the wave system and, in all save the most extreme circumstances the only effect of this component of the pressure distribution is to produce an insignificant modification to the phase speed of the waves. The surface-pressure distribution p_0 in (2.3) can accordingly be taken as p_1 given by (2.4).

The free surface-boundary conditions can be expressed as an expansion about the mean surface level, thus

$$\frac{\partial \xi}{\partial t} + \left\{\frac{\partial \xi}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z}\right\}_{z=0} + \xi\left[\frac{\partial}{\partial z}\left\{\frac{\partial \xi}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial z}\right\}\right]_{z=0} + \ldots = 0, \tag{2.5}$$

$$\frac{p_0}{\rho} + g\xi + \left(\tfrac{1}{2}u^2 + \frac{\partial \phi}{\partial t}\right)_{z=0} + \xi\left[\frac{\partial}{\partial z}\left\{\tfrac{1}{2}u^2 + \frac{\partial \phi}{\partial t}\right\}\right]_{z=0} + \ldots = 0. \tag{2.6}$$

Also, the velocity potential ϕ and the surface displacement ξ can be expanded in terms of an ordering parameter ϵ, of the order of the wave slope. Thus

$$\left.\begin{aligned} \phi &= \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots, \\ \xi &= \epsilon\xi_1 + \epsilon^2\xi_2 + \ldots. \end{aligned}\right\} \tag{2.7}$$

The first-order velocity potential ϕ_1 satisfies (2.1), together with the linearized forms of the boundary conditions (2.5) and (2.6) obtained by neglecting terms of the second and higher orders in ϵ. Therefore if (2.4) is substituted for p_0, we have

$$\left.\begin{aligned} \frac{\partial \xi_1}{\partial t} - \frac{\partial \phi_1}{\partial z} &= 0, \\ \gamma c^2 \frac{\partial \xi_1}{\partial x} + g\xi_1 + \frac{\partial \phi_1}{\partial t} &= 0, \end{aligned}\right\} \quad \text{at} \quad z = 0. \tag{2.8}$$

On differentiation of the second of these conditions with respect to t and substitution from the first, we have

$$\frac{\partial^2\phi_1}{\partial t^2}+\gamma c^2\frac{\partial}{\partial x}\left(\frac{\partial\phi_1}{\partial z}\right)+g\frac{\partial\phi_1}{\partial z}=0, \tag{2.9}$$

when $z=0$. The solution subject to this boundary condition that represents two wave components can be expressed as

$$\phi_1 = B'\exp\{k'z\}\exp\{i(k'x-\sigma't)\}+B''\exp\{k''z\}\exp\{i(k''x-\sigma''t)\}, \tag{2.10}$$

where the (complex) wave-numbers and frequencies are related by

$$\left.\begin{aligned}\sigma'^2 &= gk'(1+i\gamma),\\ \sigma''^2 &= gk'',\end{aligned}\right\} \tag{2.11}$$

where, in the present situation, $k' \gg k''$ and γ represents the growth rate of the shorter waves.

Now if the shorter waves are losing energy at the crests of the larger ones through the development of an instability with consequent splashing or breaking, the amplitude of the short waves leaving the crest will be less than those arriving at it. The growth of the shorter waves takes place from one crest to the next, and the short-wave amplitude is constant at points moving with the phase velocity c'' of the long waves. When $x=c''t$, then, $(k'x-\sigma't)$ is purely real, or

$$\operatorname{Im}(k'c''-\sigma')=0. \tag{2.12}$$

The short-wave part of the solution (2.10) is then only piecewise valid, that is, in finite stretches between one long-wave crest and the next. If $k'=k'_r+ik'_i$ (where $k'_i \ll k'_r$ since $\gamma \ll 1$), and likewise $\sigma'=\sigma'_r+i\sigma'_i$, it can be shown after a little algebra that (2.12) requires

$$\left.\begin{aligned}k'_i &= \frac{\gamma\sigma'_r}{2c''-c'},\\ \sigma'_i &= c''k'_i,\end{aligned}\right\} \tag{2.13}$$

where $c'=\sigma'_r/k'_r$ is the phase velocity of the shorter waves. The first-order velocity potential can thus be represented in real form by

$$\phi_1 = A'\exp\{k'z-k'_i(x-c''t)\}\cos\{k'_rx-\sigma'_rt+k'_iz\} + A''\exp(k''z)\cos(k''x-\sigma''t), \tag{2.14}$$

where σ'' and k'' are real. The first-order surface displacement ξ_1 is given by (2.8) as

$$\xi_1 = a'\exp\{-k'_i(x-c''t)\}\sin(k'_rx-\sigma'_rt+\epsilon')+a''\sin(k''x-\sigma''t), \tag{2.15}$$

where

$$\left.\begin{aligned}a' = -A'k'_r/\sigma'_r,\quad a'' &= -A''k''/\sigma'',\\ \epsilon' &= -\gamma(c''-c')/(2c''-c'),\end{aligned}\right\} \tag{2.16}$$

the small phase difference ϵ' being a consequence of the rate of short-wave growth.

From these expressions, the second approximation can be constructed readily using the technique given by Longuet-Higgins & Stewart (1960). From (2.1)

$$\nabla^2\phi_2 = 0, \tag{2.17}$$

and from (2.5) and (2.6), when $z = 0$,

$$\left.\begin{aligned}\frac{\partial \xi_2}{\partial t} - \frac{\partial \phi_2}{\partial z} + \frac{\partial \xi_1}{\partial x}\frac{\partial \phi_1}{\partial x} - \xi_1 \frac{\partial^2 \phi_1}{\partial z^2} &= 0,\\ g\xi_2 + \frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}(\nabla\phi_1)^2 + \xi_1 \frac{\partial^2 \phi_1}{\partial z\,\partial t} &= 0.\end{aligned}\right\} \tag{2.18}$$

Elimination of ξ_2 from these two gives after a little algebra that

$$\left\{\frac{\partial^2 \phi_2}{\partial t^2} + g\frac{\partial \phi_2}{\partial z}\right\}_{z=0} = -\left\{\frac{\partial}{\partial t}(\nabla\phi_1)^2 + \xi_1 \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z}\right)\right\}_{z=0} \tag{2.19}$$

When the first-order solutions (2.14) and (2.15) are substituted into the right-hand side of (2.19) it is found that the last group of terms has the same form as the first term, but is smaller by a factor of order γ, so that (2.19) assumes the form

$$\left\{\frac{\partial^2 \phi_2}{\partial t^2} + g\frac{\partial \phi_2}{\partial t}\right\}_{z=0} = -2A'A''k'_r k''(\sigma'_r - \sigma'')\{1 - O(\gamma)\}e^{\theta}\sin\{(k'_r - k'')\,x - (\sigma'_r - \sigma'')\,t\}, \tag{2.20}$$

where

$$\theta = -k'_i(x - c''t). \tag{2.21}$$

Equation (2.17) and the boundary condition (2.20) are satisfied by

$$\phi_2 = -\frac{A'A''k'_r k''}{\sigma''}(1 + O(\gamma))\exp\{(k'_r - k'')\,z + \theta\}\sin\{(k'_r - k'')\,x - (\sigma'_r - \sigma'')\,t\}. \tag{2.22}$$

The free surface displacement is given by

$$g\xi_2 = -\left\{\frac{\partial \phi_2}{\partial t} + \tfrac{1}{2}(\nabla\phi_1)^2 + \xi_1 \frac{\partial^2 \phi_1}{\partial z\,\partial t}\right\}_{z=0},$$

whence, on making the substitutions and writing $\chi' = k'_r x - \sigma'_r t$, $\chi'' = k''x - \sigma'' t$, we have

$$\begin{aligned}\xi_2 = -\tfrac{1}{2}a'^2 e^{2\theta} k' \cos 2\chi' - \tfrac{1}{2}a''^2 k'' \cos 2\chi''&\\ - a'a''\, e^{\theta}\{k' \cos\chi' \cos\chi'' - k'' \sin\chi' \sin\chi''\}.&\end{aligned} \tag{2.23}$$

This expression is similar in form to the one given by Longuet-Higgins & Stewart (1960) for the interaction between two steady wave trains, except that the local first-order short-wave amplitude $a'e^{\theta}$ replaces their constant value a'. This is not at all unexpected and indeed might almost have been written down *ab initio*, though the development above gives a clear idea of the approximations involved. The first two terms clearly represent the second-order distortion of the original two wave components and the third their mutual interaction with which we are concerned.

The short wave, under the influence of the wind and the longer wave, therefore assumes the form

$$\begin{aligned}\xi' &= a'\, e^{\theta} \sin\chi' - a'a''\, e^{\theta}\{k' \cos\chi' \cos\chi'' - k'' \sin\chi' \sin\chi''\}\\ &= a'\, e^{\theta} \sin\chi'\{1 + a''k'' \sin\chi''\} - a'\, e^{\theta} \cos\chi'\{k'a'' \cos\chi''\},\end{aligned} \tag{2.24}$$

which represents a wave with local wave-number

$$k = k'(1 + k''a'' \sin\chi''), \tag{2.25}$$

and local amplitude

$$a = a' e^{\theta} (1 + k''a'' \sin\chi''). \tag{2.26}$$

These formal solutions represent the development to second order of the short waves for all x and t. At the long-wave crests, the amplitude of the short waves becomes large and splashing or breaking may occur. These solutions are then of course invalid locally, though they will give a good approximation over the rest of the long-wave cycle. If the breaking zone is short and occupies only a small fraction of a wavelength near the long-wave crest, the solutions are valid over most of the cycle, from the end of the breaking zone on one crest to the beginning of this zone on the next.

The distance between the long-wave crests is specified by a change of 2π in χ'', or from (2.21), by a change of $-2\pi k'_i/k''$ in θ. Thus, if

$$\Gamma = \frac{k'_i}{k''} = \frac{k'_r}{k''}\frac{\gamma c'}{2c''-c'}, \tag{2.27}$$

the increase in short-wave amplitude from one crest to the next is

$$\delta a = 2\pi\Gamma a'(1 + k''a'') \tag{2.28}$$

to the first order in Γ, or equivalently, in γ times the number of short waves between successive crests of the long ones.

This result, (2.28), seems quite significant. It shows that the increase in short-wave amplitude from one crest to the next is the *same as if* the initial amplitude were $a'(1+k''a'')$ and the long waves were not present. But the local wave amplitude in the troughs is diminished by the divergence of the flow in the long waves, with a consequently diminished rate of growth under the wind action, yet the final wave amplitude is the same as it would have been if the initial wave amplitude had steadily increased under the wind action. In terms of the short-wave energy, this suggests that the increase in short-wave energy over the cycle (and, with local breaking of this kind, the dissipation at the crests of the long waves) is only partially the result of the wind action, and there must also be an energy loss from the long waves.

This, in fact, can be shown directly from these solutions. At the long-wave crests, the short waves experience a decrease in the effective gravitational field because of the downwards acceleration of the free surface as the long waves pass. Longuet-Higgins & Stewart (1960) have shown that the local wave energy density is given in terms of the local amplitude a by

$$E = \tfrac{1}{2}\rho g a^2\left(1 + \frac{f}{2g}\right), \tag{2.29}$$

where f is the (upwards) acceleration of the free surface. The long waves produce an acceleration $f = -a''k''g$ at the crests, so that

$$E = \tfrac{1}{2}\rho g a^2(1 - \tfrac{1}{2}a''k''). \tag{2.30}$$

If $a_0 = a'(1+k''a'')$ is the short-wave amplitude at the first crest just after the breaking zone and E_0 is the corresponding short-wave energy density, then the change in energy density between this point and a point just before the breaking zone at the next long-wave crest is given by

$$\frac{\delta E}{E_0} = \frac{2\delta a}{a_0} = 4\pi\Gamma, \tag{2.31}$$

from (2.28). But from (2.30),

$$\begin{aligned} E_0 &= \tfrac{1}{2}\rho g a_0^2(1-\tfrac{1}{2}k''a'') \\ &= \tfrac{1}{2}\rho g a'^2(1+\tfrac{3}{2}k''a'') \\ &= \bar{E}(1+\tfrac{3}{2}k''a''), \end{aligned} \tag{2.32}$$

where $\bar{E} = \frac{1}{2}\rho g a'^2$ is the mean short-wave energy density. Thus

$$\delta E = 4\pi\Gamma\bar{E}(1+\tfrac{3}{2}k''a''). \tag{2.33}$$

On the other hand, the energy supplied from the wind to a wave packet of the short waves is given by

$$\delta E_w = \int_0^T p\dot{\xi}\,dt,$$

where $T = \lambda''/(c''-\frac{1}{2}c')$ is the time taken for a wave packet to travel from one crest to the next. Substituting from (2.4), (2.25) and (2.26), it is found that, to the same order as (2.33),†

$$\delta E_w = 4\pi\Gamma\bar{E}. \tag{2.34}$$

The difference between δE and δE_w must represent the net energy exchange by non-linear interaction between the long waves and the short ones over the time interval between the departure of a short-wave packet from one crest and its arrival at the next; thus

$$\delta E_I = \delta E - \delta E_w = 6\pi\Gamma k''a''\bar{E}. \tag{2.35}$$

3. A physical interpretation

These results can be given a simple physical interpretation in terms of the radiation stress discovered by Longuet-Higgins & Stewart (1960). In the absence of short-wave growth, they showed that the short-wave energy density is a maximum, $\bar{E}(1+\frac{3}{2}k''a'')$, at the long-wave crests and a minimum, $\bar{E}(1-\frac{3}{2}k''a'')$ at the troughs. As a short-wave packet leaves a long-wave crest, it loses energy partly by the convection associated with the flow divergence in the long wave and partly (as they discovered) by losing energy to the long waves through working against the radiation stress S, which in deep water is equal to half the short-wave energy density E. As the short waves move towards the next crest, they re-acquire the same amount of energy from the long wave, so that there is no *net* energy transfer from one component to the other. This is in accordance with a general result (Phillips 1960) that continuing energy transfer from one wave component to another cannot take place at second order.

† Surface pressure components associated with the second-order surface displacements give a fourth-order contribution to δE_w.

However, if the short waves are growing under wind action, and losing energy at the long-wave crests, the energy density at points where the short waves have just left a crest is *less* than the density at points symmetrically in front of the crest. Consequently, the energy transferred from the short waves behind a crest is *less* than the energy they acquire as they approach the next one, and over the cycle there is a net energy loss from the long waves and a gain to the short. However, in a steady state, this (together with the energy acquired from the wind) is lost to the wave motion when the short waves break, so that the net effect of the process is not an energy flux from one component to another, but a dissipation of energy from the long waves.

This concept of the radiation stress can be used to provide a simple alternative derivation of the results of § 2. The short-wave energy equation takes the form

$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial x}[E(c_g + U)] + U\frac{\partial S}{\partial x} + \gamma\sigma' E, \tag{3.1}$$

where c_g is the group velocity of the short waves, U the surface velocity induced by the long waves and $S = \frac{1}{2}E$ the radiation stress. This expresses the rate of change of short-wave energy density in terms of the flow convection, the rate of working against the radiation stress and the rate of energy input from the wind. Since the motion is progressive with velocity c'', the operator $\partial/\partial x$ can be replaced by $-(c'')^{-1}\,\partial/\partial t$, and the rate of energy transfer from the wind can, to sufficient accuracy, be replaced by its mean value $\gamma\sigma'\bar{E}$. Further, the radiation stress term $U\partial S/\partial x$ can be approximated by $\partial(SU)/\partial x$, so that equation (3.1) now becomes

$$\frac{\partial}{\partial t}\{E[c'' - \tfrac{1}{2}c' - \tfrac{3}{2}U]\} - \gamma c''\sigma'\bar{E} = 0, \tag{3.2}$$

whence

$$E(c'' - \tfrac{1}{2}c' - \tfrac{3}{2}U) - E_0(c'' - \tfrac{1}{2}c' - \tfrac{3}{2}U_m) = \gamma c''\sigma'\bar{E}t, \tag{3.3}$$

where $t = 0$ marks the passage of a long-wave crest where

$$E = E_0 \quad \text{and} \quad U = U_m = k''a''c''.$$

After the passage of one long wave, $t = 2\pi/\sigma''$, $U = U_m$ again and $E = E_0 + \delta E$. Thus

$$\begin{aligned} \delta E &= \frac{2\pi\gamma c''\sigma'\bar{E}}{\sigma''(c'' - \frac{1}{2}c' - \frac{3}{2}U_m)} \\ &= 2\pi\bar{E}\,\frac{\gamma\sigma'}{k''(c'' - \frac{1}{2}c')}\{1 + \tfrac{3}{2}k''a''\} \\ &= 4\pi\Gamma\bar{E}(1 + 3k''a''), \end{aligned} \tag{3.4}$$

from (2.27) and since $k' \gg k''$. This is in accord with (2.33).

The interaction between the long waves and the short is represented by the term

$$\frac{\partial}{\partial t}\left(\frac{3EU}{2c''}\right)$$

in (3.2). The net energy transfer to the short waves over a cycle of the long ones is the integral of this, or

$$\begin{aligned} \delta E_I &= \tfrac{3}{2}U_m\delta E/c'' \\ &= 6\pi\Gamma\bar{E}k''a'', \end{aligned} \tag{3.5}$$

to the lowest order, using (3.4). This provides a simple alternative derivation of the result (2.35).

The analysis of § 2 can be extended readily to describe the interaction between two wave trains whose directions of propagation are separated by an angle θ. It is found that the short-wave energy loss across the breaking zone at the long-wave crests is given by

$$\delta E = 4\pi\Gamma\bar{E}\{1+(\tfrac{3}{2}\cos^2\theta+\sin^2\theta)\,k''a''\}, \tag{3.6}$$

and that the net energy transfer from the long waves to the short ones over a cycle is

$$\begin{aligned}\delta E_I &= 2\pi\Gamma\bar{E}k''a''(3\cos^2\theta+2\sin^2\theta)\\ &= 6\pi\Gamma\bar{E}k''a''(1-\tfrac{1}{3}\sin^2\theta),\end{aligned} \tag{3.7}$$

where

$$\Gamma = \frac{\gamma\sigma'}{k''(2c''-c'\cos\theta)}, \tag{3.8}$$

the original definition (2.27) being clearly the special case $\theta = 0$.

These last two expressions can also be found by the methods of this section. In deep water the radiation stress tensor is simply

$$\mathbf{S} = \begin{pmatrix}\tfrac{1}{2}E & 0\\ 0 & 0\end{pmatrix} \tag{3.9}$$

so that only the component of the long-wave orbit velocity in the direction of short-wave propagation is involved in the radiation stress interaction. However, the normal component convects the short-wave energy; and the energy equation corresponding to (3.1) takes the form

$$\frac{\partial E}{\partial t} = -\frac{\partial}{\partial x}[E(c_g+\tfrac{3}{2}U)]-\frac{\partial}{\partial y}EV+\gamma\sigma'E, \tag{3.10}$$

where

$$\left.\begin{aligned}U &= k''a''c''\sin\chi''\cos\theta,\\ V &= k''a''c''\sin\chi''\sin\theta.\end{aligned}\right\} \tag{3.11}$$

But

$$\frac{\partial}{\partial x} = -\frac{\cos\theta}{c''}\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial y} = -\frac{\sin\theta}{c''}\frac{\partial}{\partial t},$$

and on substitution into (3.10) and integration over a cycle of the long waves, the change in energy density at the long-wave crest is found to be

$$\delta E = 2\pi\bar{E}\frac{\gamma\sigma'}{k''(c''-\tfrac{1}{2}c'\cos\theta)}\{1+(\tfrac{3}{2}\cos^2\theta+\sin^2\theta)\,k''a''\}, \tag{3.12}$$

as in (3.6). The net energy loss from the long waves over a cycle is, in like manner, seen to be as given by (3.7).

This mechanism of energy conversion to shorter waves and its subsequent loss has some similarities to the one involved in the generation of capillary waves ahead of a sharp crest of short-gravity waves. This latter effect, which has long been known,† was analysed recently by Longuet-Higgins (1963) who showed that the transfer of energy to capillaries at the sharp crest and its subsequent rapid dissipation represents a mechanism for energy loss from the gravity waves that

† But which has sometimes been believed to have something to do with the air flow near the sharp crests.

can be many times more effective than the direct viscous dissipation. In the present problem, we have similarly an energy transfer to a smaller-scale wave motion and its subsequent loss there. It seems likely that the capillary waves limit the growth of short-gravity waves on small ponds where this phenomenon is commonly observed, and perhaps also provides dissipation from the short-gravity wave components in the open sea. The mechanism discussed here involves the action of short wind-generated waves on longer ones, which need not be particularly steep. Its most direct application seems to be in the damping of swell by locally generated waves, and this is discussed in the next section.

4. The attenuation of swell

Every mariner knows that swell dies under an opposing wind; it is a part of the sea-faring folk lore. Even a sceptic must give *some* credence to a qualitative generalization like this, based as it is on intimate acquaintance and long observation. Yet it has been difficult (for the present author, at any rate) to see how the direct action of the wind could produce an attenuation. The mechanisms that can account for the growth of waves under the influence of the wind do not work in reverse. They can provide energy to a wave travelling in the same direction as the wind but they cannot extract it from a wave moving against the wind. A kind of sheltering effect has been suggested, involving separation of the air flow in the lee of the swell, but this separation seems most implausible on physical grounds. However, the results of the previous sections do provide a way in which energy is extracted from the swell—indirectly though, through the intermediate action of the locally generated wind waves.

Consider, then, a single component of swell passing through a region of shorter waves generated by a local wind. Near the crests of the swell, the wind waves will, as soon as they are sufficiently developed, lose energy by breaking or splashing, or developing capillaries. Some of this energy is acquired from the swell, the energy δE_I transferred per cycle per unit area from the swell and subsequently lost to the wave motion being given by (3.7). The time taken for successive crests of the swell to overtake a group of the short waves is

$$\lambda''/(c'' - \tfrac{1}{2}c' \cos\theta).$$

Thus if $E_s = \frac{1}{2}\rho g a''^2$ is the energy density of the swell,

$$\begin{aligned} \frac{dE_s}{dt} &= -\frac{\delta E_I(c'' - \frac{1}{2}c'\cos\theta)}{\lambda''} \\ &= -\tfrac{3}{2}\gamma\sigma'\bar{E}k''a''(1 - \tfrac{1}{3}\sin^2\theta). \end{aligned} \tag{4.1}$$

Therefore the rate of decrease of the amplitude of the swell

$$\frac{da''}{dt} = -\frac{3\gamma\sigma'\bar{E}k''}{2\rho g}(1 - \tfrac{1}{3}\sin^2\theta) \tag{4.2}$$

is constant as long as the short-wave breaking at the crests continues. Under steady conditions, the spatial attenuation is specified by

$$\frac{da''}{dx} = -\frac{3\gamma\sigma'\bar{E}}{\rho c''^3}(1 - \tfrac{1}{3}\sin^2\theta), \tag{4.3}$$

where $c'' = (g/k)^{\frac{1}{2}}$ is the phase velocity of the swell.

This linear attenuation of wave amplitude with distance (or time) is rather unusual, though the reasons for it are quite simple. The rate of energy loss from the swell results from its working against a given short-wave radiation stress, and so is proportional to its own orbital speed. The rate of energy loss is thus proportional to the wave amplitude, and the rate of decrease in amplitude constant.

These results can be expressed alternatively by noting that $\gamma\sigma'\bar{E}$ represents the mean rate of energy transfer per unit area from the wind to the short waves. Denoting this by ϵ, we have

$$\frac{da''}{dt} = -\frac{3\epsilon}{2\rho c''^2}(1-\tfrac{1}{3}\sin^2\theta), \tag{4.4}$$

and

$$\frac{da''}{dx} = -\frac{3\epsilon}{\rho c''^3}(1-\tfrac{1}{3}\sin^2\theta). \tag{4.5}$$

A simplification, which involves little loss in accuracy, is to neglect the small directional variation in the last factor of each of these expressions, so that we have simply

$$\frac{da''}{dt} = -\frac{3\epsilon}{2\rho c''^2}, \quad \frac{da''}{dx} = -\frac{3\epsilon}{\rho c''^3}. \tag{4.6}$$

It is noteworthy that these last expressions can be applied directly to the attenuation of swell by a whole spectrum of shorter wind-generated waves, where ϵ is the total rate of energy transfer per unit area from wind to waves. Alternative expressions can be derived from (4.2) and (4.3) directly by integration over all wave-numbers or frequencies, but (4.6) is simpler and likely to be more accurate since it does not require evaluation of γ as a function of wave-number from the stability theory and the subsequent integrations over the spectrum. The total energy transfer rate ϵ can be estimated much more directly, a good approximation being given by

$$\epsilon = \rho_a u_*^2 c, \tag{4.7}$$

where ρ_a is the air density, u_* the friction velocity of the wind (so that the wind stress $\tau = \rho_a u_*^2$) and c the phase velocity of the wind waves at the spectral peak. This expression assumes that all the stress supplied by the wind appears initially as wave momentum, and is transferred to mean currents by wave breaking. Stewart (1961) suggests on empirical grounds that this is likely, and it can in fact be demonstrated theoretically using the results of Miles (1957) that this is probably so as soon as the mean square slope exceeds a very small value. With (4.7), then, we have finally the attenuation rates given simply by

$$\frac{da''}{dt} = -\frac{3\rho_a u_*^2 c}{2\rho_w c''^2}, \tag{4.8}$$

$$\frac{da''}{dx} = -\frac{3\rho_a u_*^2 c}{\rho_w c''^3}, \tag{4.9}$$

where, to avoid confusion, ρ_w denotes the water density.

To give some indication of the rate of attenuation of swell in this way some numerical examples are illuminating. Consider swell passing through the trade winds, where the wind speed is about 20 knots, so that $u_* \sim 1$ m/sec. If the local sea has a characteristic period of order 3 sec, $c = 5\,\text{m/sec}$. Then swell with a period of 20 sec decreases in amplitude by attenuation about 50 cm/1000 km of travel, while a 15 sec wave decreases 1 m/1000 km. For a 7 sec swell, (4.9) gives an attenuation of 10 m/1000 km, though the accuracy of this last figure may not be high since the condition $k' \gg k''$ is only weakly satisfied.

It is not yet possible to compare these attenuation rates with direct observational evidence, though this may soon be possible. A remarkable series of observations by Munk, Miller, Snodgrass & Barber (1963) on low-frequency waves generated in Antarctic seas, propagating through the trade wind zones and incident on Southern California have suggested that waves with periods greater than about 15 sec are attenuated only slightly (apart from the geometrical spreading) whereas those with periods less than this appear to be attenuated considerably. The energy losses seem to be consistent with those resulting from interactions of this kind in the region of the trade winds, though the wave spectra in the storm area were not measured but estimated using the observed wind field in the generating area and an empirical spectrum.

REFERENCES

BENJAMIN, T. BROOKE 1959 *J. Fluid Mech.* **6**, 161–205.
HASSELMANN, K. 1962 *J. Fluid Mech.* **12**, 481–500.
HASSELMANN, K. 1963 *J. Fluid Mech.* **15**, 385–98.
LIGHTHILL, M. J. 1962 *J. Fluid Mech.* **14**, 385–98.
LONGUET-HIGGINS, M. S. 1953 *Phil. Trans.* A, **245**, 535–81.
LONGUET-HIGGINS, M. S. 1963 *J. Fluid Mech.* (in the Press).
LONGUET-HIGGINS, M. S. & STEWART, R. W. 1960 *J. Fluid Mech.* **8**, 565–83.
MILES, J. W. 1957 *J. Fluid Mech.* **3**, 185–204.
MILES, J. W. 1960 *J. Fluid Mech.* **7**, 469–78.
MUNK, W. H., MILLER, G., SNODGRASS, F. & BARBER, N. 1963 *Phil. Trans.* A, (in the Press).
PHILLIPS, O. M. 1960 *J. Fluid Mech.* **9**, 193–217.
PHILLIPS, O. M. & KATZ, E. J. 1961 *J. Mar. Res.* **19**, 57–69.
STEWART, R. W. 1961 *J. Fluid Mech.* **10**, 189–94.